

# Regularized path integrals and anomalies

## - U(1) chiral gauge theory -

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### Abstract

We analyse the origin of the Adler-Bell-Jackiw (ABJ) anomaly of chiral U(1) gauge theory within the framework of regularized path integrals. Momentum or position space regulators allow for mathematically well-defined path integrals but violate local gauge symmetry. It is known how (nonanomalous) gauge symmetry can be recovered in the renormalized theory in this case [1]. Here we analyse U(1) chiral gauge theory to show how the appearance of anomalies manifests itself in such a context. We show that the three-photon amplitude leads to a violation of the Slavnov-Taylor-Identities which cannot be restored on taking the UV limit in the renormalized theory. We point out that this fact is related to the nonanalyticity of this amplitude in the infrared region.

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## 1 Introduction

When analysing a quantum field theory model one typically starts from a lagrangian encoding its field content and symmetries. Still, writing a lagrangian generally does not define the theory, not even when restricting to perturbation theory. This is due in particular to the need of renormalization which requires to modify the lagrangian by adding counter terms, or, in the language of the Wilson renormalization group [29], to follow the flow of the relevant parameters of the theory. It may then turn out that the process of renormalization does not fully respect the symmetry structure of the initial lagrangian. If a symmetry is inevitably broken by the quantum corrections, one talks of an anomalous symmetry. It may also happen that the symmetry is only broken at an intermediate stage through regulators which make the theory well-defined and can be recovered, once these regulators are taken away again. It is generally admitted that theories which can be fully regularized without breaking any of their symmetries, cannot be anomalous.

In this paper we want to come back to the chiral  $U(1)$  gauge theory - which is known to be anomalous [2], [4], [10], [5], [31] - in a momentum space regularization scheme, which breaks gauge invariance from the beginning. Such regularizations are used when establishing the differential flow equations [28] of the renormalization group [29], which allow for an elegant inductive approach to perturbative renormalization theory [26].

Most often perturbative renormalization of gauge theories is performed with the aid of dimensional regularization which at first sight respects local gauge symmetry. Most of the work and in particular most of the calculations have been done in this scheme ever since it has been known to exist. For chiral gauge theories containing the four-dimensional Levi-Civita tensor  $\varepsilon_{\mu\nu\rho\sigma}$ , dimensional regularization does not fully respect the gauge symmetry however, since this tensor does not have a straightforward generalization

to  $4 + \varepsilon$  dimensions. In spite of its great advantages the dimensional scheme also has some drawbacks, mainly on the mathematical side<sup>1</sup>. It not only defies to be given rigorous meaning in path integral formulations, it does not even directly apply in a mathematical sense to perturbative Green functions as a whole without splitting them into graphs. Thus, in some sense it is farthest away from nonperturbative analysis.

On the other hand analysis of symmetries and functional relations in field theory are largely based on path integral formulations. It therefore seems to be important to study gauge theories in the rigorous framework of regularized path integrals on which the flow equations are based. A proof of perturbative renormalizability of spontaneously broken  $SU(2)$ -Yang-Mills theory with the aid of flow equation was performed in [1]. In [18] an analysis of QED with massive photons was performed. Its extension to massless photons in [20] became technically quite involved and could (should) be improved nowadays. A fully rigorous analysis of QCD in this framework, including the infrared part of the problem, still has to be performed.

Let us shortly comment on the strategy of proof of [1]. The (ultraviolet) power counting part of the flow equation renormalization proof is universal and simple for all renormalizable theories. For gauge theories we have to show that gauge invariance can be restored when the cutoffs are taken away. On the level of the Green functions (which are not gauge invariant) this means that we have to verify the Slavnov-Taylor identities (STI) of the theory. They allow to argue that physical quantities such as the S-matrix are gauge-invariant [30]. On analysing the flow equations (FE) for a gauge theory one realizes that the restoration of the STI depends on the choice of the renormalization conditions chosen and is not true in general. More precisely, since gauge invariance is violated in the regularized theory, the renormalization group flow will generally produce nonvanishing contributions to all those relevant parameters of the theory, which are forbidden by gauge invariance. The question is then: Can we use the freedom in adjusting the renormalization conditions such that the STI are nevertheless restored in the end? To answer this question a first observation is crucial: The violation of the STI in the regularized theory can be expressed through Green functions carrying an operator insertion, which depends on the regulators. FE theory for such insertions tells us that these Green functions will vanish once the cutoffs are removed, if we achieve renormalization conditions on the noninserted Green functions such that the inserted ones, which are calculated from those, have vanishing renormalization conditions for all relevant terms, i.e. up to the dimension of the insertion (which turns out to be 5). In case of spontaneously broken Yang-Mills theory as well as for QED it could then be shown that there exist classes of renormalization conditions such that the relevant part of the STI vanishes, and in consequence such that the STI are restored after taking away the cutoffs.

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<sup>1</sup>for example when calculating the three-photon-amplitude analysed in App. A in the dimensional scheme, it is often stated that this amplitude or its derivatives are *arbitrary* in some sense. Thus one may ask oneself in which sense and at which stage the starting point is well-defined mathematically ; as it is e.g. in a momentum-space regularized version of the theory.

In the present paper we want to analyse the mechanism behind the appearance of the anomaly in chiral  $U(1)$  gauge theory in this framework. As a consequence of the previous remarks an anomaly should manifest itself through the appearance of a finite relevant contribution to the STI which cannot be eliminated by a suitable choice of renormalization conditions. Our analysis reveals that this appearance is closely related to the infrared divergences of the chiral gauge theory. In fact it will turn out that complete Bose symmetry together with analyticity around zero momentum - which would hold in fully massive theories - would prevent the appearance of the ABJ anomaly. The deduction of the anomalous Ward or Slavnov-Taylor identities proceeds in the same way as in the  $SU(2)$ -case. There is no room for a contribution from the integration measure, which seems to be in contrast with the deduction of the anomaly by Fujikawa [13], [14]. In this respect, we discuss the Jacobian of regularized BRS-transformations, and we also discuss Fujikawa's argument. We note that recently chiral anomalies have also been analysed nonperturbatively in two-dimensional models like the Thirring model [24], [9]. Conceptually this approach is close to ours since it is also based on regularized path integrals, which in this case can be analysed constructively, i.e. beyond perturbation theory.

Our paper is organized as follows. In section 2 we introduce the classical action of the chiral  $U(1)$  gauge theory, its symmetries and the abelian BRST-transformations [8], [27]. In section 3 we introduce regularized path integrals, certain concepts from FE theory, and we recall the statements on renormalizability we need. In particular we introduce the above mentioned operator insertions. When using the FE it is natural to analyse the generating functional of free propagator amputated Schwinger functions. The analysis of the STI is however technically simpler for one-particle irreducible vertex functions so that we introduce the generating functionals of both, together with the corresponding renormalizability statements. In section 4 we derive the violated Slavnov-Taylor identities (VSTI) for the regularized theory emphasizing the terms related to the anomaly. Using explicit results on the regularized three-photon-amplitude we show that *the STI cannot be restored in the UV limit for any choice of renormalization conditions*. As regards the general aspects of path integral analysis we try to keep the presentation in sections 3 and 4 short, referring to the more detailed analogous deductions presented in [1] in the technically more involved nonabelian case.

In the appendices we analyse the ABJ anomaly in the regularized theory and reveal its relation to the infrared singularity of the massless fermion chiral gauge theory (App. A); we show that for straightforward regularizations the Jacobian associated with the BRS-transformation in the path integral equals 1 (App. B); and we shortly comment on Fujikawa's argument (App. C).

## 2 The classical action of chiral $U(1)$ gauge theory

We consider the axial-vector-coupling abelian gauge theory with fermions and massive axial-vector gauge bosons, in euclidean signature. We will mainly restrict to massless fermions. The classical action has the form

$$S_{inv} = \int dx \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} (i\partial^\mu + gA^\mu) \psi \right\} , \quad \text{with} \quad \int dx \equiv \int_{\mathbb{R}^4} d^4x . \quad (1)$$

The field strength tensor is defined as

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) . \quad (2)$$

The coupling parameter  $g$  is real. The Euclidean Dirac matrices verify the anticommutation relations  $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ , and we adopt the convention

$$\gamma_5 \equiv -\gamma_0\gamma_1\gamma_2\gamma_3 ,$$

such that  $\gamma_5^2 = 1$ . For massless fermions the action (1) is invariant under local gauge transformations of the fields

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu u(x), \quad \psi(x) \rightarrow e^{igu(x)\gamma_5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{igu(x)\gamma_5} \quad (3)$$

with  $u : \mathbf{R}^4 \rightarrow \mathbf{R}$ , smooth. Mass terms for fermions are excluded by global (chiral) gauge symmetry.

Aiming at a quantized theory, pure gauge degrees of freedom have to be eliminated. We choose the standard covariant gauge fixing with  $\alpha \in \mathbf{R}_+$ , and we also introduce a mass  $M > 0$  for the gauge field, and thus add the following contribution to the action

$$S_{g.f.} = \int dx \left\{ \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \frac{M^2}{2} A_\mu A_\mu \right\} . \quad (4)$$

With regard to functional integration this condition is implemented by introducing (bosonic but) anticommuting<sup>2</sup> Faddeev-Popov ghost and antighost fields  $c$  and  $\bar{c}$  [11], [10], respectively, and forming with these scalar fields the additional term in the action

$$S_{gh} = \int dx \bar{c} (\partial_\mu \partial_\mu - \alpha M^2) c . \quad (5)$$

Hence, we have the total "classical action"

$$S_{BRS} = S_{inv} + S_{g.f.} + S_{gh} , \quad (6)$$

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<sup>2</sup>the fields  $\psi, \bar{\psi}, c, \bar{c}$  anticommute among each other

which is decomposed as

$$S_{BRS} = \int dx \{ \mathcal{L}_{\text{quad}}(x) + \mathcal{L}_{\text{int}}(x) \} \quad (7)$$

into its quadratic part

$$\mathcal{L}_{\text{quad}} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \frac{1}{2} M^2 A_\mu A_\mu + \bar{\psi} i\partial^\mu \psi - \bar{c}(-\Delta + \alpha M^2)c, \quad (8)$$

where<sup>3</sup>  $\Delta \equiv \partial_\mu \partial_\mu$ , and into its interaction part

$$\mathcal{L}_{\text{int}} = g \bar{\psi} \not{A} \gamma_5 \psi. \quad (9)$$

We impose the following transformation properties of the fields under the discrete symmetries of charge conjugation  $C$  and parity  $P$  :

$$\begin{aligned} A_\mu &\xrightarrow{C} A_\mu, \quad \psi \xrightarrow{C} \psi^c = i\gamma_2 \gamma_0 \bar{\psi}^T, \quad \bar{\psi} \xrightarrow{C} i\psi^T \gamma_0 \gamma_2, \quad c \xrightarrow{C} c, \quad \bar{c} \xrightarrow{C} \bar{c}, \\ A_0(x) &\xrightarrow{P} -A_0(\tilde{x}), \quad A_i(x) \xrightarrow{P} A_i(\tilde{x}), \quad \psi(x) \xrightarrow{P} \eta \gamma_0 \psi(\tilde{x}), \quad \bar{\psi} \xrightarrow{P} \eta^* \bar{\psi}(\tilde{x}) \gamma_0, \\ c(x) &\xrightarrow{P} -c(\tilde{x}), \quad \bar{c}(x) \xrightarrow{P} -\bar{c}(\tilde{x}). \end{aligned}$$

Here  $\eta$  is an undetermined phase factor, and we set  $\tilde{x} \equiv (x_0, -\vec{x})$ . Note in particular that  $A_\mu$  transforms as an axial vector.

As a prerequisite to state the symmetries of  $S_{BRS}$  (7), composite classical fields are introduced as follows:

$$\rho_\mu = \partial_\mu c, \quad \rho^j = ig(\gamma_5 \psi)^j c, \quad \bar{\rho}^j = ig(\bar{\psi} \gamma_5)^j c. \quad (10)$$

The classical action  $S_{BRS}$  (7) then shows the following symmetries :

- i) Euclidean invariance:  $S_{BRS}$  is an  $O(4)$ -scalar.
- ii) Charge conjugation invariance.
- iii) BRS-invariance:

The BRS-transformations of the basic fields are defined as

$$\begin{aligned} A_\mu(x) &\longrightarrow A_\mu(x) - \rho_\mu(x) \varepsilon, \\ \psi^j(x) &\longrightarrow \psi^j(x) - \rho^j(x) \varepsilon, \\ \bar{\psi}^j(x) &\longrightarrow \bar{\psi}^j(x) - \bar{\rho}^j(x) \varepsilon, \\ c(x) &\longrightarrow c(x), \\ \bar{c}(x) &\longrightarrow \bar{c}(x) - \frac{1}{\alpha} \partial_\nu A_\nu(x) \varepsilon, \end{aligned} \quad (11)$$

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<sup>3</sup>we use the summation convention

using the composite fields (10);  $\varepsilon$  is a Grassmann element not depending on space-time that commutes with the fields  $A_\mu$  but anticommutes with the (anti-)fermions  $\psi, \bar{\psi}$  and the (anti-)ghosts  $c, \bar{c}$ .

To show the BRS-invariance of the total classical action (6) one first observes that the composite classical fields (10) are themselves invariant under the BRS-transformations (11). Herewith it follows easily that the sum  $S_{\text{g.f.}} + S_{\text{gh}}$  is invariant under the transformation (11). Finally, on  $S_{\text{inv}}$  act only the BRS-transformations of the fields  $A_\mu, \psi, \bar{\psi}$ , which amounts to local gauge transformations.

We observe that upon scaling the composite fields (11) entering the BRS-transformations as well as  $S_{\text{gh}}$  (5), by a factor of  $\lambda$ , the corresponding  $S_{\text{BRS}}$  remains invariant under such BRS-transformations. BRS-invariance is considered to be sufficient for the gauge invariance of the S-matrix if it exists [30]. Note that contrary to electrodynamics, charge conjugation invariance does not forbid terms which are odd monomials in the gauge field. The absence of such terms in QED is often termed Furry's theorem.

The fields  $A_\mu, \bar{c}, c$  and  $\bar{\psi}, \psi$  have mass dimensions 1 and  $3/2$  respectively. We associate the ghost number 1 to  $c$ , ghost number  $-1$  to  $\bar{c}$ , and ghost number 0 to  $A_\mu, \bar{\psi}, \psi$ . With these assignments the action has mass dimension 0 and ghost number 0.

### 3 Regularized path integrals and renormalization

In this section we shall introduce the path integral formulation of chiral U(1) gauge theory. From momentum space regularized path integrals one derives the flow equations of the renormalization group on which renormalization theory in full generality can be based. In fact the flow equations allow to deduce inductive bounds on the Schwinger functions which imply renormalizability, as was realized by Polchinski [26], see also [21]. We try to be short on renormalization theory here since it is our aim to confirm that the theory we consider *cannot be renormalized maintaining local gauge symmetry*. Using flow equations it is straightforward to see that it *can be renormalized abandoning local gauge symmetry*, a fact which one might state as *renormalizability in the weak sense* [22]. Thus we will not present the flow equations here, but just introduce the formalism on which they are based and from which we can deduce the Slavnov-Taylor-identities (STI), which are violated in the presence of cutoffs. We will then present the statements on renormalization theory we need in order to be able to verify whether the STI can be restored on taking away the regulators. A complete presentation of renormalization theory, in a case where the answer to this question is affirmative, was presented in [1].

### 3.1 The regularized effective action

Bosonic field variables are generically denoted by  $\phi$ . Generally one may consider that they are smooth functions<sup>4</sup>. We will use the following concise notations :

$$\langle \phi, \phi' \rangle \equiv \int dx \phi(x) \phi'(x), \quad (\phi * \phi')(y) \equiv \int dx \phi(x) \phi'(x - y).$$

As regards Fourier transforms we set

$$\begin{aligned} \phi(x) &= \int_p e^{ipx} \hat{\phi}(p) \quad \text{with} \quad \int_p \equiv \int \frac{d^4 p}{(2\pi)^4} \quad \Longleftrightarrow \quad \hat{\phi}(p) = \int dx e^{-ipx} \phi(x) \\ \Rightarrow \quad \frac{\delta}{\delta \phi(x)} &= (2\pi)^4 \int_p e^{-ipx} \frac{\delta}{\delta \hat{\phi}(p)}. \end{aligned}$$

Quantization of the theory by means of functional integration in the realm of (formal) power series is based on a Gaussian measure related to the quadratic part (8) of  $S_{\text{BRS}}$  (7). Denoting the differential operators appearing with the various fields as

$$D_{\mu\nu} := (-\Delta + M^2) \delta_{\mu\nu} - \frac{1-\alpha}{\alpha} \partial_\mu \partial_\nu, \quad i \not{\partial}_{ij} := i \partial_\mu (\gamma_\mu)_{ij}, \quad D := -\Delta + \alpha M^2, \quad (12)$$

we write

$$\int dx \mathcal{L}_{\text{quad}}(x) = \frac{1}{2} \langle A_\mu^a, D_{\mu\nu} A_\nu^a \rangle + \langle \bar{\psi}, i \not{\partial} \psi \rangle - \langle \bar{c}, D c \rangle. \quad (13)$$

To the differential operators (12) are associated the (free) propagators

$$C_{\mu\nu}(x, y) = \int_k e^{ik(x-y)} C_{\mu\nu}(k), \quad S_{ij}(x, y) = \int_k e^{ik(x-y)} S_{ij}(k), \quad C(x, y) = \int_k e^{ik(x-y)} C(k) \quad (14)$$

with

$$C_{\mu\nu}(k) = \frac{1}{k^2 + M^2} \left( \delta_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2 + \alpha M^2} \right), \quad S_{ij}(k) = \frac{\not{k}_{ij}}{k^2}, \quad C(k) = \frac{1}{k^2 + \alpha M^2}. \quad (15)$$

The Gaussian product measure is then defined with the aid of covariances which are a regularized version of the propagators (14), (15). We choose a cutoff function  $\sigma_\Lambda(k^2)$  and set

$$\sigma_{\Lambda, \Lambda_0}(k^2) \equiv \sigma_{\Lambda_0}(k^2) - \sigma_\Lambda(k^2). \quad (16)$$

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<sup>4</sup>the support of Gaussian measures depends on their regularity properties. For efficient regulators our assumption turns out to be almost surely realized. For weaker forms of regulators the subsequent expressions are still well-defined in the support of the measure even if it exceeds the space of  $C^\infty$ -functions. We do not make explicit a finite-volume cutoff since our statements on vertex and Schwinger functions hold in the infinite volume limit. It is straightforward to start by considering the theory on a torus, see [25].

For the bosons we may for example choose as in [1]

$$\sigma_\Lambda^B(k^2) = \exp\left(-\frac{(k^2 + M^2)(k^2 + \alpha M^2)(k^2)^2}{\Lambda^8}\right). \quad (17)$$

This cutoff function is positive, invertible and analytic, and has the property

$$\frac{d}{dk^2} \sigma_\Lambda^B(k^2)|_{k^2=0} = 0, \quad (18)$$

which is helpful in the analysis of the STI in [1]. For the fermions we choose a weaker cutoff more adapted for explicit 1-loop calculations of section A.2. In fact we simply choose

$$\sigma_\Lambda^F(k^2) = \left(1 - \frac{k^2}{k^2 + \Lambda^2}\right) \quad (19)$$

or higher powers thereof, i.e. a Pauli-Villars type cutoff.

Employing these cutoff functions we define the regularized propagators, with UV-cutoff  $\Lambda_0 < \infty$  and a flow parameter  $\Lambda$  satisfying  $0 \leq \Lambda \leq \Lambda_0$ ,

$$\begin{aligned} C_{\mu\nu}^{\Lambda, \Lambda_0}(k) &\equiv C_{\mu\nu}(k) \sigma_{\Lambda, \Lambda_0}^B(k^2), & C^{\Lambda, \Lambda_0}(k) &\equiv C(k) \sigma_{\Lambda, \Lambda_0}^B(k^2), \\ S^{\Lambda, \Lambda_0}(k) &\equiv S(k) \sigma_{\Lambda, \Lambda_0}^F(k^2). \end{aligned} \quad (20)$$

It is convenient to introduce a short collective notation for the various fields and their sources:

i) We denote the physical fields and the corresponding sources, respectively, by

$$\varphi = (A_\mu, \psi^j, \bar{\psi}^j), \quad J = (j_\mu, \bar{\chi}^j, \chi^j), \quad (21)$$

ii) and all fields and their respective sources by

$$\Phi = (\varphi, c, \bar{c}), \quad K = (J, \bar{\eta}, \eta). \quad (22)$$

The sources  $\chi^j$ ,  $\bar{\chi}^j$  and  $\eta$ ,  $\bar{\eta}$  are Grassmann elements,  $\eta$ ,  $\bar{\eta}$  have ghost number +1 and -1, respectively. In the sequel, we exclusively use left derivatives with respect to these quantities.

The characteristic functional of the Gaussian product measure with the covariances from (20), (15) - multiplied by  $\hbar$  in view of the loop expansion - is then given by

$$\int d\mu_{\Lambda, \Lambda_0}(\Phi) e^{\frac{1}{\hbar} \langle \Phi, K \rangle} = e^{\frac{1}{\hbar} P^{\Lambda, \Lambda_0}(K)}, \quad (23)$$

where

$$\varphi(x)J(x) \equiv A_\mu(x)j_\mu(x) + \bar{\chi}^j(x)\psi^j(x) + \bar{\psi}^j(x)\chi^j(x), \quad (24)$$

and

$$\langle \varphi, J \rangle \equiv \int dx \varphi(x) J(x) , \quad \langle \Phi, K \rangle \equiv \langle \varphi, J \rangle + \int dx \left( \bar{c}(x) \eta(x) + \bar{\eta}(x) c(x) \right) , \quad (25)$$

$$P^{\Lambda, \Lambda_0}(K) = \frac{1}{2} \langle j_\mu, C_{\mu\nu}^{\Lambda, \Lambda_0} j_\nu \rangle + \langle \bar{\chi}, S^{\Lambda, \Lambda_0} \chi \rangle - \langle \bar{\eta}, C^{\Lambda, \Lambda_0} \eta \rangle . \quad (26)$$

We now consider the generating functional  $L^{\Lambda, \Lambda_0}(\Phi)$  of the regularized (through  $\sigma_{\Lambda, \Lambda_0}$ ) connected amputated Schwinger functions (CAS) given by

$$e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\Phi) + I^{\Lambda, \Lambda_0})} = \int d\mu_{\Lambda, \Lambda_0}(\Phi') e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\Phi' + \Phi)} . \quad (27)$$

We impose  $L^{\Lambda, \Lambda_0}(0) = 0$  so that the constant  $I^{\Lambda, \Lambda_0}$  is the vacuum part which is proportional to the volume because of translation invariance. It therefore requires to consider the theory at first in a finite volume  $\Omega \subset \mathbf{R}^4$ . For details see [25].

Since the regularization necessarily violates the local gauge symmetry, the bare functional  $L^0(\Phi) = L^{\Lambda_0, \Lambda_0}(\Phi)$  in a first stage has to be chosen sufficiently general in order to allow for a finite limit  $\Lambda_0 \rightarrow \infty$  at the end. We set

$$L^0(\Phi) = L^{\Lambda_0, \Lambda_0}(\Phi) = \int dx \mathcal{L}_{\text{int}}(x) + L_{c.t.}^{\Lambda_0, \Lambda_0}(\Phi) \quad (28)$$

thus adding to the interaction part (9) of classical origin, counter terms  $L_{c.t.}^{\Lambda_0, \Lambda_0}$ , which a priori include all local terms of mass dimension  $\leq 4$  permitted by the unbroken *global* symmetries, i.e. Euclidean  $O(4)$ -invariance, charge conjugation and global gauge invariance (3). There are six such terms, by definition all at least of order  $\mathcal{O}(\hbar)$ . The general bare functional can be written as follows :

$$\begin{aligned} L_{c.t.}^{\Lambda_0, \Lambda_0} = & \int d^4x \left[ (\bar{\psi} \psi)^0 i \bar{\psi} \not{\partial} \psi + \frac{(\delta M^2)^0}{2} A^2 + \frac{\Sigma_{\text{long}}^0}{2\alpha} (\partial A)^2 + \frac{\Sigma_{\text{trans}}^0}{4} F^2 \right. \\ & \left. + \frac{(F^{AAAA})^0}{4!} (A^2)^2 + (\delta g)^0 \bar{\psi} \not{A} \gamma_5 \psi \right] . \end{aligned} \quad (29)$$

In the abelian theory the ghosts are not coupled to the other fields. It is therefore not necessary to introduce counter terms for the ghost fields. Note that a fermion mass term is not compatible with global gauge symmetry.

We also note that for  $\Lambda = \Lambda_0$  (i.e. when the regularized propagator vanishes), we have the intuitively obvious equality between the generating functionals of the connected and one-particle irreducible functions [1] denoted by  $\Gamma$

$$L_{c.t.}^{\Lambda_0, \Lambda_0} = \Gamma_{c.t.}^{\Lambda_0, \Lambda_0} . \quad (30)$$

## 3.2 Inserted Schwinger functions

To analyse the Slavnov-Taylor identities (STI), we have to consider Schwinger functions with a composite field inserted, too. Two kinds of such insertions have to be dealt with: local insertions implementing the BRS-variations, and a space-time integrated insertion representing the violation of the STI.

The classical composite BRS-fields (10) have mass dimensions 2 and 5/2 (the latter if a fermion field appears). They transform as axial vector, spinor and anti-spinor respectively, and they have fermion number 0,  $\pm 1$  and ghost number 1. Hence, allowing for counterterms, we introduce the bare composite fields<sup>5</sup>

$$\rho_\mu^0(x) = R_1^0 \partial_\mu c(x) , \quad (31a)$$

$$\rho^{j,0}(x) = R_2^0 i g (\gamma_5 \psi(x))^j c(x) , \quad (31b)$$

$$\bar{\rho}^{j,0}(x) = R_3^0 i g (\bar{\psi}(x) \gamma_5)^j c(x) , \quad (31c)$$

keeping the notation from (10) but using it henceforth exclusively according to (31a)-(31c). We set

$$R_i^0 = 1 + \mathcal{O}(\hbar) , \quad (32)$$

thus viewing the counterterms again as formal power series in  $\hbar$ ; the tree order  $\hbar^0$  provides the classical terms (10). We note that the modified composite fields (31a)-(31c) remain *invariant* under the BRS-transformations (11) if we employ the generalized composite fields (31a)-(31c) in place of the original ones, (10). Contrarily to the nonabelian case, this invariance does not enforce additional constraints on the  $R_i^0$ .

To generate Schwinger functions with such insertions, the bare interaction (28) is modified adding the composite fields (31a)-(31c) coupled to corresponding sources

$$\tilde{L}_0 = \tilde{L}^{\Lambda_0, \Lambda_0}(\rho; \Phi) \equiv L^{\Lambda_0, \Lambda_0}(\Phi) + L^{\Lambda_0, \Lambda_0}(\rho) , \quad (33)$$

$$L^{\Lambda_0, \Lambda_0}(\rho) = \int dx \{ \zeta_\mu(x) \rho_\mu^0(x) + \bar{\zeta}^j(x) \rho^{j,0}(x) + \bar{\rho}^{j,0}(x) \zeta^j(x) \} . \quad (34)$$

According to the properties of these composite fields, the sources  $\zeta_\mu$ ,  $\bar{\zeta}^j$ ,  $\zeta^j$  are Grassmann elements, they have canonical dimension 2 respectively 3/2 for the last two, and ghost number  $-1$ . For the insertions and their respective sources we also introduce a short collective notation

$$\rho = (\rho_\mu, \rho^j, \bar{\rho}^j) , \quad \zeta = (\zeta_\mu, \bar{\zeta}^j, \zeta^j) . \quad (35)$$

Using now (33) in place of  $L^{\Lambda_0, \Lambda_0}$  as the bare action in the representation (27), provides the functional  $\tilde{L}^{\Lambda, \Lambda_0}(\rho; \Phi)$ , from which the generating functional of the regularized CAS

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<sup>5</sup>one may ask whether one should also introduce a factor of  $R_4^0$  for the BRS-transform of the antighost, cf. the last relation in (10); such a factor is redundant however because we may always choose an overall normalization freely.

with one insertion  $\rho_\mu^0(x)$  follows as

$$L_{\zeta_\mu}^{\Lambda, \Lambda_0}(x; \Phi) \equiv \frac{\delta}{\delta \zeta_\mu(x)} \tilde{L}^{\Lambda, \Lambda_0}(\rho; \Phi)|_{\rho=0}, \quad (36)$$

and similarly for the other insertions from (34). In the infinite volume limit, and performing a Fourier transform of the insertion position we obtain

$$\hat{L}_{\zeta_\mu}^{\Lambda, \Lambda_0}(q; \Phi) = \int dx e^{iqx} L_{\zeta_\mu}^{\Lambda, \Lambda_0}(x; \Phi). \quad (37)$$

We shall describe in Section 4, how the initial regularization, necessarily violating the STI, leads to another insertion which we denote as

$$\tilde{L}^{\Lambda_0, \Lambda_0}(\theta) \equiv \int dx \theta(x) N(x), \quad N(x) = Q(x) + Q'(x; \Lambda_0^{-1}). \quad (38)$$

Here  $\theta$  is another source function. The individual terms of  $N(x)$  involve at most five fields and have ghost number 1. Furthermore,  $Q(x)$  is a local polynomial in the fields and their derivatives, having canonical mass dimension  $D = 5$ , whereas  $Q'(x; \Lambda_0^{-1})$  is nonpolynomial in the field momenta but suppressed by powers of  $\Lambda_0^{-1}$ . In fact we will only need the spacetime integrated insertion which is obtained from the local one via functional derivation and subsequent integration. We denote

$$L_\theta^{\Lambda, \Lambda_0}(\Phi) \equiv \int dx L_\theta^{\Lambda, \Lambda_0}(x; \Phi) \equiv \int dx \frac{\delta}{\delta \theta(x)} \tilde{L}^{\Lambda, \Lambda_0}(\theta; \Phi)|_{\theta=0}. \quad (39)$$

### 3.3 Proper Vertex Functions

Our analysis of the STI will be based on a representation in terms of proper vertex functions (1PI), since the extraction of relevant parts from the STI is simpler and more transparent in terms of those than in terms of the CAS. We will basically skip here the passage to the 1PI-functionals which is performed explicitly in [1], [25], and only give some basic results.

The field variables of the Legendre transformed functional are denoted through underlined variables  $\underline{A}_\mu, \underline{\psi}^j, \underline{\bar{\psi}}^j, \underline{c}, \underline{\bar{c}}$ , and analogously for the collective notations  $\underline{\varphi}, \underline{\Phi}$ . We can then obtain the generating functional of regularized vertex functions

$$\Gamma^{\Lambda, \Lambda_0}(\underline{\Phi}),$$

and also the corresponding generating functional of inserted regularized vertex functions

$$\tilde{\Gamma}^{\Lambda, \Lambda_0}(\rho; \underline{\Phi}).$$

Since we restrict to perturbation theory, the generating functional will be considered within a formal loop expansion

$$\Gamma^{\Lambda, \Lambda_0}(\underline{\Phi}) = \sum_{l=0}^{\infty} \hbar^l \Gamma_l^{\Lambda, \Lambda_0}(\underline{\Phi}) . \quad (40)$$

Furthermore, decomposing into particular  $n$ -point vertex functions we introduce a multiindex  $n$ , the components of which denote the number of each source field species appearing, together with its modulus and its norm defined as follows :

$$n = (n_A, n_\psi, n_{\bar{\psi}}, n_c, n_{\bar{c}}), |n| = n_A + n_\psi + n_{\bar{\psi}} + n_c + n_{\bar{c}}, ||n|| = n_A + \frac{3}{2}(n_\psi + n_{\bar{\psi}}) + n_c + n_{\bar{c}} . \quad (41)$$

The corresponding regularized vertex functions in momentum space are then obtained through functional derivation

$$(2\pi)^{4(|n|-1)} \delta_{\underline{\Phi}(p)}^n \Gamma_l^{\Lambda, \Lambda_0}(\underline{\Phi})|_{\underline{\Phi} \equiv 0} = \delta(p_1 + \dots + p_{|n|}) \Gamma_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_{|n|}) , \quad (42)$$

$$(2\pi)^{4(|n|-1)} \delta_{\underline{\Phi}(p)}^n \Gamma_{\zeta_\mu; l}^{\Lambda, \Lambda_0}(q; \underline{\Phi})|_{\underline{\Phi} \equiv 0} = \delta(q + p_1 + \dots + p_{|n|}) \Gamma_{\zeta_\mu; l,n}^{\Lambda, \Lambda_0}(q; p_1, \dots, p_{|n|}) , \quad (43)$$

$$(2\pi)^{4(|n|-1)} \delta_{\underline{\Phi}(p)}^n \Gamma_{\theta; l}^{\Lambda, \Lambda_0}(\underline{\Phi})|_{\underline{\Phi} \equiv 0} = \delta(p_1 + \dots + p_{|n|}) \Gamma_{\theta; l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_{|n|}) . \quad (44)$$

For the sake of a slim appearance, the notation does not reveal how the momenta are assigned to the multiindex  $n$ , and in addition, the  $O(4)$ -tensor structure remains hidden. By definition the  $n$ -point function is completely symmetric (antisymmetric) if the variables that belong to each of the commuting (anti-commuting) species occurring are permuted.

### 3.4 Weak renormalizability

In this section we report on a number of results obtained from renormalization theory based on flow equations, which we will need subsequently in the analysis of the STI. We try to be short in this respect since it will turn out (as expected) that the model considered cannot be renormalized as a gauge theory.

With the aid of the flow equations one can deduce inductive bounds on the Schwinger functions which imply renormalizability, as was realized by Polchinski [26], see also [21], and [17] where the flow equations for composite operators were introduced. For a more recent presentation see [25]. The facts necessary to treat theories with massless fields can be inferred from [16], see also [19], [18], and [22].

As usual the relevant parameters of the theory have to be fixed through renormalization conditions. The relevant part of the functional  $\Gamma^{0, \Lambda_0}$  is analysed in 4.2.1. In a (partially) massless theory marginal terms which are (logarithmically) infrared divergent

by power-counting at zero momentum, have to be renormalized at non-exceptional<sup>6</sup> external momenta. We therefore impose the following renormalization conditions at any loop order  $l \in \mathbb{N}$

$$(\Gamma_{l,(0,2)}^{0,\Lambda_0})_{ij}(0) = 0, \quad (45)$$

$$\partial_\mu(\Gamma_{l,(0,2)}^{0,\Lambda_0})_{ij}(0) = \Sigma_l^{\bar{\psi}\psi} (\gamma_\mu)_{ij}, \quad (46)$$

$$(\Gamma_{l,(2,0)}^{0,\Lambda_0})_{\mu\nu}(p_R) = (\delta M^2)_l \delta_{\mu\nu} + (p_R^2 \delta_{\mu\nu} - p_{R,\mu} p_{R,\nu}) \Sigma_{trans,l} + \frac{p_{R,\mu} p_{R,\nu}}{\alpha} \Sigma_{long,l}, \quad (47)$$

$$(\Gamma_{l,(3,0)}^{0,\Lambda_0})_{\mu\nu\rho}(0) = 0, \quad (48)$$

$$(\Gamma_{l,(4,0)}^{0,\Lambda_0})_{\mu\nu\rho\sigma}(\vec{p}_R^{(4)}) = \frac{F_l^{AAAA}}{3} (\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}), \quad (49)$$

$$(\Gamma_{l,(1,2)}^{0,\Lambda_0})_{\mu ij}(0) = (\delta g)_l (\gamma_\mu \gamma_5)_{ij}. \quad (50)$$

In (46) we derive with respect to the momentum associated to the field  $\bar{\psi}$ . We denote by  $p_R = p_{1R}$  a fixed nonvanishing momentum. Then the four momenta in  $\vec{p}_R^{(4)} = (p_{1R}^{(4)}, p_{2R}^{(4)}, p_{3R}^{(4)}, p_{4R}^{(4)})$  may be chosen such that they point from the centre into the corners of a tetrahedron - or similarly into the corners of an equilateral triangle in the case of the momenta  $\vec{p}_R^{(3)}$  of a three point function. From power counting one may also expect that  $\partial_\sigma(\Gamma_{l,(3,0)}^{0,\Lambda_0})_{\mu\nu\rho}(p_1, p_2, p_3)$  contains a relevant contribution, which then should be proportional to the tensor  $\varepsilon_{\mu\nu\rho\sigma}$ . In fact the analysis of this term in section A, see in particular (92), (94) excludes such a contribution. Still this term is directly related to the anomaly in the STI, see below (75, 76), (86, 87) and section A. For inserted vertex functions, with  $D$  being the dimension of the insertion, similarly all local terms of dimension  $\leq D$  have to be fixed by renormalization conditions, where analogous restrictions on the external momenta have to be observed. For the inserted functional  $\Gamma_1$  appearing in the VSTI we have  $D = 5$ , and the corresponding relevant terms are listed explicitly in section 4.2.2. With these renormalization conditions the subsequent proposition holds for non-inserted vertex functions, if we start from the (inter)action (28), where the counter terms are calculated as functions of the renormalization conditions. For inserted vertex functions it holds with the same conditions imposed on the noninserted theory, and for a bare inserted functional calculated as before from analogous renormalization conditions on the relevant inserted terms. We state the proposition without proof, since its proof can be inferred from [1], [16], [22]; knowing that the anomaly will prevent us from making the corresponding statement on strong renormalizability (i.e. including the restoration of gauge symmetry) anyway.

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<sup>6</sup>i.e. no nontrivial subsum vanishes

**Proposition 3.1. Weak renormalizability of chiral  $U(1)$  gauge theory**

For fixed non-exceptional external momentum configurations  $\vec{p}$  the vertex functions

$$\Gamma_{l,n}^{\Lambda,\Lambda_0}(\vec{p}) \quad (51)$$

are uniformly bounded in  $\Lambda_0$ . Furthermore the limits

$$\lim_{\Lambda_0 \rightarrow \infty} \Gamma_{l,n}^{\Lambda,\Lambda_0}(\vec{p}) \equiv \Gamma_{l,n}^{\Lambda}(\vec{p})$$

and

$$\lim_{\Lambda \rightarrow 0} \Gamma_{l,n}^{\Lambda}(\vec{p}) \equiv \Gamma_{l,n}(\vec{p})$$

exist and are smooth functions in the open set of non-exceptional momenta.

The same statements also hold for inserted vertex functions

$$\Gamma_{\zeta_\mu; l,n}^{\Lambda,\Lambda_0}(q; \vec{p}) \quad (52)$$

and

$$\Gamma_{\theta; l,n}^{\Lambda,\Lambda_0}(\vec{p}) . \quad (53)$$

It is also possible to control the singularities of the vertex functions at exceptional momenta, see [16].

For the analysis of the possible restitutio of the STI in the renormalized theory the following statement on the inserted functions  $\Gamma_{1;l,n}^{\Lambda,\Lambda_0}(\vec{p})$  is important (see [1])

**Proposition 3.2. Restitution theorem**

If all renormalization constants imposed on the relevant part of  $\Gamma_{\theta;l}^{0,\Lambda_0}$  vanish and if possibly nonvanishing irrelevant contributions to the bare functional  $\Gamma_{\theta;l}^{\Lambda_0,\Lambda_0}$  are bounded by  $O\left(\Lambda_0^{D-||n||-|w|} \mathcal{P}(\log(\frac{\Lambda_0}{\mu}))\right)$  - for a suitable mass scale  $\mu > 0$  - then for non-exceptional momenta the inserted functions

$$\Gamma_{\theta;l,n}^{0,\Lambda_0}(\vec{p}) \quad (54)$$

vanish in the limit  $\Lambda_0 \rightarrow \infty$ , at least as  $O\left(\Lambda_0^{-1} \mathcal{P} \log\left(\frac{\Lambda_0}{\mu}\right)\right)$ .

The polynomials  $\mathcal{P}$  have nonnegative coefficients which may depend on  $l, n, \mu, g, \alpha$ , but not on  $\Lambda, \Lambda_0$ .

Again we do not give a proof of this statement. In fact *the presence of the anomaly turns out to be an obstruction of its application on chiral  $U(1)$  gauge theory*.

## 4 The Violated Slavnov-Taylor identities

### 4.1 Deduction of the VSTI from the path integral

To examine the violation of the STI produced by the UV cutoff  $\Lambda_0$  we proceed in analogy with [1]. We start from the generating functional of the regularized Schwinger functions at the value  $\Lambda = 0$  of the flow parameter <sup>7</sup>,

$$Z^{0,\Lambda_0}(K) = \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L^{\Lambda_0,\Lambda_0}(\Phi) + \frac{1}{\hbar}\langle\Phi, K\rangle}. \quad (55)$$

The Gaussian measure  $d\mu_{0,\Lambda_0}(\Phi)$  corresponds to the quadratic form  $\frac{1}{\hbar}Q^{0,\Lambda_0}(\Phi)$ , cf. (23),

$$Q^{0,\Lambda_0}(\Phi) = \frac{1}{2}\langle A_\mu, (C^{0,\Lambda_0})_{\mu\nu}^{-1}A_\nu\rangle + \langle\bar{\psi}, (S^{0,\Lambda_0})^{-1}\psi\rangle - \langle\bar{c}, (C^{0,\Lambda_0})^{-1}c\rangle. \quad (56)$$

We define *regularized* BRS-variations (11), (31a)-(31c) of the fields by

$$\delta_{BRS} \varphi(x) = -(\sigma_{0,\Lambda_0} * \rho)(x) \varepsilon, \quad (57)$$

$$\delta_{BRS} \bar{c}(x) = -(\sigma_{0,\Lambda_0} * \frac{1}{\alpha} \partial_\nu A_\nu)(x) \varepsilon. \quad (58)$$

The BRS-variation of the Gaussian measure has the form

$$d\mu_{0,\Lambda_0}(\Phi) \mapsto d\mu_{0,\Lambda_0}(\Phi) \left( 1 - \frac{1}{\hbar} \delta_{BRS} Q^{0,\Lambda_0}(\Phi) \right). \quad (59)$$

This BRS-variation of the Gaussian measure is obtained in the same way as for SU(2)-gauge theory, under the hypothesis that there is no Jacobian stemming from the redefinition of the field variables themselves. This is justified by

**Lemma 4.1.** We introduce a cube of side length  $L$  in  $\mathbb{R}^4$  and expand the field variables in plane wave modes, imposing periodic boundary conditions. We introduce an UV cutoff  $\Lambda_0$  and restrict to wave numbers  $k_n \in \mathbb{R}^4$  such that  $|k_n| \leq \Lambda_0$ . Imposing these regularizations the Jacobian associated with the change of variables (57, 58) equals 1.

The elementary proof of this statement is in App. B. From the proof it is quite evident that the statement holds for larger classes of regulators and mode expansions. This is in some sense opposed to the deduction of the anomaly by Fujikawa [13], [14] who relates it to a nontrivial Jacobian. On the other hand a statement analogous to ours can be found [15], sect. II.A. We comment on Fujikawa's argument in App. C.

Inspecting (56) we observe that the factor  $\sigma_{0,\Lambda_0}$  of the BRS-variations (57, 58) just cancels its inverse entering the inverted propagators. Hence, the BRS-variation of the

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<sup>7</sup>again one should stay in finite volume as long as the vacuum part is involved

Gaussian measure has mass dimension  $D = 5$ . Invariance of the regularized generating functional  $Z^{0,\Lambda_0}(K)$ , (55) under the BRS-variations (57, 58) then provides the *violated Slavnov-Taylor identities*

$$0 \stackrel{!}{=} \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\hbar}L^{\Lambda_0,\Lambda_0}(\Phi)+\frac{1}{\hbar}\langle\Phi,K\rangle} \left( \delta_{BRS} \langle\Phi, K\rangle - \delta_{BRS} (Q^{0,\Lambda_0} + L^{\Lambda_0,\Lambda_0}) \right). \quad (60)$$

The BRS-variations appearing in (60) can be dealt with, considering corresponding modified generating functionals, where the notations are chosen as in 3.2:

i) The modified bare interaction (33) is defined

$$\tilde{Z}^{0,\Lambda_0}(K, \rho) \equiv \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\hbar}\tilde{L}^{\Lambda_0,\Lambda_0}(\rho; \Phi)+\frac{1}{\hbar}\langle\Phi,K\rangle}. \quad (61)$$

ii) The BRS-variations of the bare action and of the Gaussian measure

$$L_\theta^{\Lambda_0,\Lambda_0} \varepsilon \equiv -\delta_{BRS} (Q^{0,\Lambda_0} + L^{\Lambda_0,\Lambda_0}) = \int dx N(x) \varepsilon \quad (62)$$

form a space-time integrated insertion with ghost number 1. The variation of  $L^{\Lambda_0,\Lambda_0}$ , however, keeps the regularizing factor  $\sigma_{0,\Lambda_0}$  of (57, 58), thus the integrand  $N(x)$  is no longer a polynomial in the fields and their derivatives. We treat the integrand  $N(x)$  as a local insertion with a source  $\theta(x)$ , cf. (39). Introducing the corresponding bare action  $\tilde{L}^{\Lambda_0,\Lambda_0}(\theta; \Phi)$ , we define the functional <sup>8</sup>  $\tilde{Z}^{0,\Lambda_0}(K, \theta)$  in analogy to (61).

In terms of these modified  $Z$ -functionals the VSTI (60) can now be written

$$\mathcal{D}_{0,\Lambda_0} \tilde{Z}^{0,\Lambda_0}(K, \rho)|_{\rho=0} = \int dx \frac{\delta}{\delta\theta(x)} \tilde{Z}^{0,\Lambda_0}(K, \theta)|_{\theta=0}. \quad (63)$$

Here we introduced a *regularized* BRS-operator<sup>9</sup>

$$\mathcal{D}_{0,\Lambda_0} = \left\langle J, \sigma_{0,\Lambda_0} \frac{\delta}{\delta\zeta} \right\rangle + \left\langle \frac{1}{\alpha} \partial_\nu \frac{\delta}{\delta j_\nu}, \sigma_{0,\Lambda_0} \eta \right\rangle. \quad (64)$$

The modified  $Z$ -functional (61) is related to the corresponding generating functional of modified CAS by <sup>10</sup>

$$\tilde{Z}^{0,\Lambda_0}(K, \rho) = e^{\frac{1}{\hbar}P^{0,\Lambda_0}(K)} e^{-\frac{1}{\hbar}(\tilde{L}^{0,\Lambda_0}(\rho; \varphi, c, \bar{c}) + I^{0,\Lambda_0})}, \quad (65)$$

and analogously in case of  $\tilde{Z}^{0,\Lambda_0}(K, \theta)$ . Starting from the relations between the generating functionals  $\tilde{Z}$  and the corresponding generating functionals of the vertex-functions we can

<sup>8</sup> Abusing notation we let the variables  $\theta$  and  $\rho$ , respectively, denote different functions.

<sup>9</sup>  $\langle J, \frac{\delta}{\delta\zeta} \rangle$  is short for  $\int dx \{ j_\mu(x) \delta_{\zeta_\mu(x)} + \bar{\chi}^j(x) \delta_{\bar{\zeta}^j(x)} - \delta_{\zeta^j(x)} \chi^j(x) \}$ .

<sup>10</sup> The vacuum part  $I^{0,\Lambda_0}$  is the same as in the case without insertion, since the insertion has ghost number 1

convert (65) at the value  $\Lambda = 0$  into the violated Slavnov-Taylor identities for proper vertex functions, on substituting there the fields  $\Phi$  by the underlined fields  $\underline{\Phi}$  which are the variables of the Legendre transform. We obtain

$$\Gamma_\theta^{0,\Lambda_0}(\underline{\varphi}, \underline{c}, \bar{\underline{c}}) = \left\langle \frac{\delta \Gamma^{0,\Lambda_0}}{\delta \underline{\varphi}}, \sigma_{0,\Lambda_0} \Gamma_\zeta^{0,\Lambda_0} \right\rangle - \left\langle \frac{1}{\alpha} \partial_\nu \underline{A}_\nu, \sigma_{0,\Lambda_0} \frac{\delta \Gamma^{0,\Lambda_0}}{\delta \bar{\underline{c}}} \right\rangle \quad (66)$$

with

$$\Gamma_\theta^{0,\Lambda_0}(\underline{\varphi}, \underline{c}, \bar{\underline{c}}) = L_1^{0,\Lambda_0}(\varphi, c, \bar{c}) . \quad (67)$$

We rewrite the VSTI (66) more explicitly as

$$\begin{aligned} \Gamma_\theta^{0,\Lambda_0} = & -\frac{1}{\alpha} \left\langle \frac{\delta \Gamma^{0,\Lambda_0}(\underline{\Phi})}{\delta \bar{\underline{c}}}, \sigma_{0,\Lambda_0} * \partial \underline{A} \right\rangle + \left\langle \sigma_{0,\Lambda_0} * \frac{\delta \Gamma^{0,\Lambda_0}(\underline{\Phi})}{\delta \underline{A}_\nu}, \Gamma_{\zeta_\nu}^{0,\Lambda_0}(\underline{\Phi}) \right\rangle \\ & - \left\langle \sigma_{0,\Lambda_0} * \frac{\delta \Gamma^{0,\Lambda_0}(\underline{\Phi})}{\delta \underline{\psi}}, \Gamma_{\bar{\zeta}}^{0,\Lambda_0}(\underline{\Phi}) \right\rangle - \left\langle \Gamma_\zeta^{0,\Lambda_0}(\underline{\Phi}), \sigma_{0,\Lambda_0} * \frac{\delta \Gamma^{0,\Lambda_0}(\underline{\Phi})}{\delta \underline{\psi}} \right\rangle . \end{aligned} \quad (68)$$

$\Gamma_\theta^{\Lambda, \Lambda_0}$  represents the violation of the STI. Due to the fact that the BRS-transform increases the dimension of a monomial in the fields by one unit,  $\Gamma_\theta$  has to be interpreted as the generating functional of 1PI-functions carrying an operator insertion of dimension 5 and ghost number one. Therefore relevant terms in this functional have mass dimension  $\leq 5$ . Still following [1] we now analyse the relevant contributions to the VSTI (68) in 4.2<sup>11</sup>.

## 4.2 The relevant contributions to the VSTI

### 4.2.1 The relevant contributions to the functional $\Gamma$

The generating functional  $\Gamma^{0,\Lambda_0}$  is invariant under the Euclidean group, under charge conjugation and under global (chiral) gauge transformations. We start listing the contributions to the relevant part of the generating functional  $\Gamma^{0,\Lambda_0}$  i.e. those terms of mass dimension  $\leq 4$  which respect these symmetries. We do not underline field variables nor do we indicate the dependence on  $\Lambda_0$  or the loop-order  $l$ .

•

$$\Gamma_2 = \int_p A_\mu(p) A_\nu(-p) \Gamma_{\mu\nu}^{AA}(p) + \bar{\psi}^i(p) \Gamma_{ij}^{\bar{\psi}\psi}(p) \psi^j(-p) - \bar{c}(p) c(-p) \Gamma^{\bar{c}c}(p)$$

---

<sup>11</sup>In [1] we also analysed the VSTI at the bare side, i.e. at  $\Lambda = \Lambda_0$ , in order to verify the corresponding boundary conditions for Proposition 3.2. Since here we will show that the anomaly prevents us from verifying the required boundary conditions at  $\Lambda = 0$ , this second step becomes obsolete.

$$\begin{aligned}
\Gamma_{\mu\nu}^{AA}(p) &= \frac{1}{2} \left[ (M^2 + \delta M^2) \delta_{\mu\nu} + (p^2 \delta_{\mu\nu} - p_\mu p_\nu) (1 + \Sigma_{trans}) + \frac{p_\mu p_\nu}{\alpha} (1 + \Sigma_{long}) \right] \\
\Gamma^{\bar{\psi}\psi}(p) &= -\not{p} (1 + \Sigma^{\bar{\psi}\psi}) \\
\Gamma^{\bar{c}c}(p) &= p^2 + \alpha M^2 .
\end{aligned}$$

Since ghosts do not interact with other fields nor with themselves, the renormalization procedure does not modify the expression of their propagator. We have

$$\Sigma^{\bar{\psi}\psi}, \Sigma_{trans}, \Sigma_{long}, \delta M^2 = O(\hbar) .$$

Due to global gauge invariance there is no mass term in  $\Gamma^{\bar{\psi}\psi}$ .

•

$$\Gamma_3 = \int_{p,q} \bar{\psi}^i(p) \Gamma_{\mu,ij}^{\bar{\psi}A\psi}(p,q) \psi^j(q) A_\mu(-p-q) + A_\mu(p) A_\nu(q) A_\rho(-p-q) \Gamma_{\mu\nu\rho}^{AA}(p,q)$$

$$\Gamma_{\mu,ij}^{\bar{\psi}A\psi}(p,q) = (\gamma_\mu \gamma_5)_{ij} F^{\bar{\psi}A\psi}, \quad F^{\bar{\psi}A\psi} = g + \delta g ,$$

with  $\delta g$ ,  $\Gamma_{\mu\nu\rho}^{AA} = O(\hbar)$ . The structure of  $\Gamma_{\mu\nu\rho}^{AA}$  is analysed in section A.

•

$$\Gamma_4 = \int_{p,q,r} \frac{1}{4!} A_\mu(p) A_\nu(q) A_\rho(r) A_\sigma(-(p+q+r)) \Gamma_{\mu\nu\rho\sigma}^{AAAA}(p,q,r) ,$$

$$\Gamma_{\mu\nu\rho\sigma}^{AAAA}(p,q,r) = \frac{F^{AAAA}}{3} (\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) ,$$

with  $F^{AAAA} = O(\hbar)$ .

#### 4.2.2 The relevant contributions to the functional $\Gamma_\theta$

Expanding  $\Gamma_\theta$  up to terms of mass dimension 5 in fields and momenta we obtain the relevant terms which are listed below. We first write the corresponding contribution to the (V)STI for the corresponding field content and then the relation which follows if one imposes the corresponding relevant part of  $\Gamma_\theta$  to vanish. This relation is expressed in terms of the momenta and of the renormalization constants. The field content is indicated in the upper index of  $\Gamma_\theta$ .

1.

$$\Gamma_\theta^{A_\mu c}(p, -p) = \left( \frac{1}{\alpha} i p_\mu \Gamma^{\bar{c}c}(p) - 2i R_1 p_\nu \Gamma_{\nu\mu}^{AA}(p) \right) \quad (69)$$

$$0 \stackrel{!}{=} i M^2 p_\mu \left( 1 - R_1 \left( 1 + \frac{\delta M^2}{M^2} \right) \right) , \quad (70)$$

$$0 \stackrel{!}{=} i p_R^2 p_{R,\mu} \frac{1}{\alpha} (1 - R_1 (1 + \Sigma_{long})) . \quad (71)$$

2.

$$\begin{aligned} \Gamma_\theta^{\bar{\psi}_i \psi_j c}(p_1, p_2, -p_1 - p_2) = & -i R_1 p_{3\mu} \Gamma_{\mu,ij}^{\bar{\psi}A\psi} - i g R_3 \left( \gamma_5 \Gamma^{\bar{\psi}\psi}(-p_2) \right)_{ij} \\ & + i g R_2 \left( \gamma_5 \Gamma^{\bar{\psi}\psi}(p_1) \right)_{ij} \end{aligned} \quad (72)$$

$$0 \stackrel{!}{=} i (\not{p}_1 \gamma_5)_{ij} \left[ R_1 (g + \delta g) - g R_2 (1 + \Sigma^{\bar{\psi}\psi}) \right] , \quad (73)$$

$$0 \stackrel{!}{=} i (\not{p}_2 \gamma_5)_{ij} \left[ R_1 (g + \delta g) - g R_3 (1 + \Sigma^{\bar{\psi}\psi}) \right] . \quad (74)$$

3. <sup>12</sup>

$$\Gamma_\theta^{A_{\mu_1} A_{\mu_2} c}(p_1, p_2, p_3) = -3! g R_1 p_{3\mu} \Gamma_{\mu\mu_1\mu_2}^{AAA}(p_3, p_1) , \quad p_3 \equiv -p_1 - p_2 \quad (75)$$

$$0 \stackrel{!}{=} \partial^w (p_{3\mu} \Gamma_{\mu\mu_1\mu_2}^{AAA}(p_3, p_1)) \Big|_{p_i \equiv p_{R,i}^{(3)}} , \quad |w| \leq 2 . \quad (76)$$

4.

$$\Gamma_\theta^{\bar{\psi}_i \psi_j A_{\mu_1} c}(p_1, p_2, p_3, p_4) = -i g (R_3 - R_2) \left( \gamma_5 \Gamma_{\mu_1}^{\bar{\psi}A\psi} \right)_{ij} , \quad p_4 = -p_1 - p_2 - p_3 \quad (77)$$

$$0 \stackrel{!}{=} g(g + \delta g) (R_3 - R_2) . \quad (78)$$

5.

$$\Gamma_\theta^{A_{\mu_1} A_{\mu_2} A_{\mu_3} c}(p_1, p_2, p_3, p_4) = -4! R_1 p_{4,\mu} \Gamma_{\mu\mu_1\mu_2\mu_3}^{AAAA}(p_1, p_2, p_3, p_4) \quad (79)$$

$$0 \stackrel{!}{=} \partial^w (p_{4,\mu} \Gamma_{\mu\mu_1\mu_2\mu_3}^{AAAA}(p_1, p_2, p_3, -p_1 - p_2 - p_3)) \Big|_{p_i \equiv p_{R,i}^{(4)}} , \quad |w| \leq 1 . \quad (80)$$

---

<sup>12</sup>We use the notation  $w \equiv (w_{1,1}, \dots, w_{n-1,4})$ ,  $w_{i,\mu} \in \mathbf{N}_0$ ,  $\partial^w \equiv \prod_{i,\mu} \left( \frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}}$ ,  $|w| \equiv \sum_{i,\mu} w_{i,\mu}$ .

The subsequent five relations on the renormalization conditions allow to verify the conditions indicated behind them

$$R_2 = R_3 \Rightarrow (78) , \quad (81)$$

$$R_1 = \frac{1}{1 + \Sigma_{long}} \Rightarrow (71) , \quad (82)$$

$$\frac{\delta M^2}{M^2} = \Sigma_{long} \Rightarrow (70) , \quad (83)$$

$$R_1 = (1 + \Sigma^{\bar{\psi}\psi}) \frac{g}{g + \delta g} R_2 \Rightarrow (73), (74) , \quad (84)$$

$$F^{AAAA} = 0 \Rightarrow (80) . \quad (85)$$

The last relation is sufficient to ensure (79) since the tensor structure of  $\Gamma_{\mu\mu_1\mu_2\mu_3}^{AAAA}$  implies that the higher order contributions in an expansion around  $\vec{p}_R^{(4)}$  are irrelevant<sup>13</sup>. A simple solution of (81) to (85) is given by imposing the value 0 for all quantities of order  $\hbar$ . The tensor structure and the one-loop contributions to  $\Gamma_{\mu\nu\rho}^{AAA}$  are analysed in section A. Subsequently we just write  $\Gamma_{\mu\nu\rho}$ . As a consequence of explicit calculation we find

**Proposition 4.1.** For  $\Lambda = 0$  and  $\Lambda_0 < \infty$  the contracted three-photon-amplitude has the Feynman parameter representation (denoting  $d\mu_5 = \prod_{i=1}^5 dx_i \delta(1 - \sum_i x_i)$  )

$$p_{1\mu} \Gamma_{\mu\nu\rho}^{0,\Lambda_0} = \frac{2}{\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} \int d\mu_5 \frac{x_3 \Lambda_0^6}{[\tilde{x}_{25} p_2^2 + \tilde{x}_3 p_3^2 + 2x_{25} x_3 p_2 \cdot p_3 + x_{123} \Lambda_0^{273}]} . \quad (86)$$

The integrand and its up to second derivatives are absolutely integrable. In the UV limit the integral converges uniformly in momentum space and is given by

$$\lim_{\Lambda_0 \rightarrow \infty} (p_{1\mu} \Gamma_{\mu\nu\rho}^{0,\Lambda_0}) = \frac{2}{\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} \int d\mu_5 \frac{x_3}{x_{123}^3} = \frac{1}{6\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} . \quad (87)$$

Furthermore, we have the bounds

$$|\partial^w (p_{1\mu} \Gamma_{\mu\nu\rho}^{0,\Lambda_0} - p_{1\mu} \Gamma_{\mu\nu\rho}^{0,\infty})| = O\left(\frac{p^{4-|w|}}{\Lambda_0^2}\right) , \quad |w| \leq 2 .$$

As a consequence the relevant part of the STI given through the r.h.s. of (75) and (76) cannot be made vanish for any choice of renormalization point. In fact, as is explained in

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<sup>13</sup>

$$\begin{aligned} \Gamma_{\mu\nu\rho\sigma}^{AAAA}(p_1, \dots, p_4) &= (\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) F^{AAAA} + p'^2 (\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) F_1 \\ &+ [p'_{1\mu} p'_{2\nu} \delta_{\rho\sigma} + p'_{1\mu} p'_{3\rho} \delta_{\nu\sigma} + p'_{1\mu} p'_{4\sigma} \delta_{\nu\rho} + p'_{2\nu} p'_{3\rho} \delta_{\mu\sigma} + p'_{2\nu} p'_{4\sigma} \delta_{\mu\rho} + p'_{3\rho} p'_{4\sigma} \delta_{\mu\nu}] F_2 \\ &+ O(p'^4) \end{aligned}$$

with  $p' = p - p_R$ .

App. A.1, there is no relevant local term corresponding to the three-photon-amplitude, and its second derivatives (at any non-exceptional momenta) do not identically vanish according to (87). For more details on the three-photon-amplitude see App. A. Our conclusion is

**Theorem 1.** The chiral  $U(1)$  gauge theory given through the Lagrangian (1) is not renormalizable in the strong sense, that is to say such that the Slavnov-Taylor-Identities are restored in the renormalized theory. The obstruction is due to a nonvanishing relevant (in the sense of the renormalization group) contribution of the three-vector-boson amplitude  $\Gamma_{\mu\nu\rho}^{AAA}$  violating these identities so that the STI violating functional (67, 68) satisfies

$$\left[ \Gamma_{\theta}^{0, \Lambda_0} \right]_{ins} \neq 0 , \quad (88)$$

for all (large) values of the UV cutoff  $\Lambda_0$ .

In fact the anomaly is closely related to the infrared singular behaviour of the (derivatives of) the three-photon-amplitude. If the amplitude were analytic around zero momentum, the anomaly could not appear, as is explained in the next section and follows from Lemma A.1.

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## A Analysis of the three-photon amplitude

### A.1 Tensor structure, symmetries

The three-(axial) photon-amplitude

$$\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$$

is a tensor w.r.t. the euclidean  $O(4)$ -group. Euclidean symmetry, the parity transformation of the axial vector field - which enforces the appearance of the Levi-Civita tensor  $\epsilon_{\alpha\beta\gamma\delta}$  - and translation invariance which implies  $p_1 + p_2 + p_3 = 0$ , permit to obtain the following decomposition of the tensor  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$  in invariants :

$$\begin{aligned} \Gamma_{\mu\nu\rho}(p_1, p_2, p_3) = & A_1(123)p_{1\tau}\epsilon_{\tau\mu\nu\rho} + A_2(123)p_{2\tau}\epsilon_{\tau\mu\nu\rho} \\ & + A_3(123)p_{1\nu}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\mu\rho} + A_4(123)p_{2\nu}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\mu\rho} + A_5(123)p_{1\mu}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\nu\rho} \\ & + A_6(123)p_{2\mu}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\nu\rho} + A_7(123)p_{1\rho}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\mu\nu} + A_8(123)p_{2\rho}p_{1\alpha}p_{2\beta}\epsilon_{\alpha\beta\mu\nu} . \end{aligned}$$

Here we use the shorthand notation  $A_i(\sigma(1)\sigma(2)\sigma(3))$  for  $A_i(p_{\sigma(1)}^2, p_{\sigma(2)}^2, p_{\sigma(3)}^2)$ , and the  $A_i$  are euclidean scalars. Using also complete Bose symmetry w.r.t. the 6 permutations of

$$(p_1, \mu), (p_2, \nu), (p_3, \rho)$$

and considering the momentum configurations, values of the tensor indices and identities between tensor components following from Bose symmetry as indicated in (89, 90, 91)

•

$$p_1 = (p_{11}, 0, p_{13}, p_{14}), \quad p_2 = (0, 0, 0, p_{24}), \quad (\mu, \nu, \rho) = (1, 1, 2), (1, 2, 1), \quad (89)$$

$$\Gamma_{\mu\nu\rho}(p_1, p_2, p_3) = \Gamma_{\nu\mu\rho}(p_2, p_1, p_3) ,$$

•

$$p_1 = (p_{11}, 0, 0, p_{14}), \quad p_2 = (p_{21}, 0, 0, p_{24}), \quad (\mu, \nu, \rho) = (1, 2, 3), \quad (90)$$

$$\Gamma_{\mu\nu\rho}(p_1, p_2, p_3) = \Gamma_{\nu\mu\rho}(p_2, p_1, p_3) = \Gamma_{\rho\nu\mu}(p_3, p_2, p_1) = \Gamma_{\mu\rho\nu}(p_1, p_3, p_2) ,$$

•

$$p_1 = (0, 0, p_{13}, p_{14}), \quad p_2 = (0, 0, p_{23}, p_{24}), \quad (\mu, \nu, \rho) = (1, 2, 3), \quad (91)$$

$$\Gamma_{\mu\nu\rho}(p_1, p_2, p_3) = \Gamma_{\nu\mu\rho}(p_2, p_1, p_3) ,$$

and solving them in terms of the eight invariants  $A_i$ , we obtain the following relations:

$$A_1(123) + A_1(231) + A_1(312) = 0 \quad (92)$$

$$A_1(123) = A_1(321)$$

$$A_2(123) = -A_1(213)$$

$$A_5(123) = -A_4(213)$$

$$A_6(123) = -A_3(213)$$

$$A_6(123) = A_8(213)$$

$$A_7(123) = A_4(231)$$

$$A_3(123) = A_4(123) - A_4(321) . \quad (93)$$

Thus  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$  can be expressed in terms of only two amplitudes, for example  $A_1 = A$ ,  $A_4 = B$ :

$$\Gamma_{\mu\nu\rho}(p_1, p_2, p_3) = [A(1, 2, 3)p_1^\tau - A(2, 1, 3)p_2^\tau] \varepsilon_{\tau\mu\nu} \quad (94)$$

$$+ [B(1, 2, 3)p_{3\nu} + B(3, 2, 1)p_{1\nu}] \epsilon_{\alpha\beta\rho\mu} p_1^\alpha p_2^\beta$$

$$+ [B(3, 1, 2)p_{2\mu} + B(2, 1, 3)p_{3\mu}] \epsilon_{\alpha\beta\nu\rho} p_1^\alpha p_2^\beta$$

$$+ [B(2, 3, 1)p_{1\rho} + B(1, 3, 2)p_{2\rho}] \epsilon_{\alpha\beta\mu\nu} p_1^\alpha p_2^\beta . \quad (95)$$

We note that under more restricted kinematical conditions (and thus with a weaker result) a similar analysis was performed in [12]. We can resume our findings in the following

**Lemma A.1.** The amplitude  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$  can be written in the form (94) where the scalar amplitudes  $A, B$  depend on the euclidean invariants  $p_1^2, p_2^2, p_3^2$  only.

If  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$  is analytic at vanishing momentum, as is the case in a fully massive theory, its dependence on  $A$  (94) excludes any relevant local contribution to  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$ .

Note that *analyticity thus would exclude the appearance of an anomaly*<sup>14</sup>. We also note that a local contribution to  $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$  compatible with the symmetries has to be at least of dimension 6. The corresponding term in the lagrangian then takes the form

$$(\partial_\rho A_\rho) \tilde{F}^{\mu\nu} F_{\mu\nu} , \quad \text{with} \quad \tilde{F}^{\mu\nu} \equiv \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} .$$

The results of Lemma A.1 are confirmed at 1-loop order by explicit calculation in A.2.

## A.2 Explicit results on the Pauli-Villars regularized 1-loop amplitude

We consider the one-loop triangular diagram (fig.1), with complete Bose symmetry between the external legs. In our case, we take an IR-regulator  $\Lambda$ . We use a Pauli-Villars regularization<sup>15</sup> of the fermionic propagators:

$$\frac{\not{k}}{k^2} \rightarrow \frac{\not{k}}{k^2} \sigma_{\Lambda, \Lambda_0}^F(k) = S^{\Lambda, \Lambda_0}(k) , \quad (96)$$

and we introduce Feynman parameters with the following notations:

$$\int d\mu_n \equiv \left( \prod_{i=1}^n \int_0^1 dx_i \right) \delta(1 - \sum_{i=1}^n x_i) , \quad (97)$$

$$x_{i_1 \dots i_m} = x_{i_1} + \dots + x_{i_m} , \quad \bar{x}_{i_1 \dots i_m} = 1 - x_{i_1 \dots i_m} , \quad \tilde{x}_{i_1 \dots i_m} = x_{i_1 \dots i_m} \bar{x}_{i_1 \dots i_m} . \quad (98)$$

1. The regularized (symmetrized) one-loop triangle diagram is given by

$$\Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0} = 2 \int_k \mathbf{tr} \left[ \gamma_5 S^{\Lambda, \Lambda_0}(k) \gamma_\nu S^{\Lambda, \Lambda_0}(k - p_2) \gamma_\mu S^{\Lambda, \Lambda_0}(k + p_3) \gamma_\rho \right] ,$$

---

<sup>14</sup>this conclusion is based on the complete Bose symmetry of the amplitude. In more complicated theories like the standard model there are fermion triangle contributions which are not fully Bose symmetric due to the presence of several vector boson species. In this case the previous conclusion does not hold.

<sup>15</sup>we hardly found any calculations in the literature which are not based on the dimensional scheme. Sometimes a global PV-regularization, obtained on introducing a heavy fermion is used [5]. If not directly applied to the integrand, this still does not lead to well-defined integrals, however.

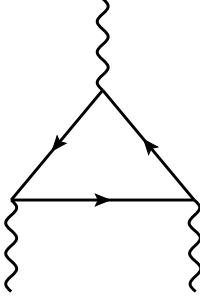


Figure 1: The triangular diagram

and at one-loop order, the amplitudes  $A$  and  $B$  introduced previously are given by:

$$\begin{aligned}
 A(1, 2, 3) = & \frac{\alpha^3}{\pi^2} \int \frac{d\mu_6}{D^3} (2x_{56} - x_{1234}) \\
 & + \frac{\alpha^3}{\pi^2} \int \frac{d\mu_6}{D^4} \left[ p_1^2 [x_{34}^2(1 - x_{34} - 3x_{12}) - x_{56}^2(1 - x_{56}) + x_{34}(x_{12} - x_{56})] \right. \\
 & \quad + p_2^2 [3x_{56}(x_{12}^2 + x_{34}^2) + 2x_{12}x_{34} - x_{56}x_{1234}] \\
 & \quad \left. + p_3^2 [x_{12}^2(1 - x_{12} - 3x_{34}) - x_{56}^2(1 - x_{56}) + x_{12}(x_{34} - x_{56})] \right]
 \end{aligned}$$

and

$$B(1, 2, 3) = \frac{2\alpha^3}{\pi^2} \int \frac{d\mu_6}{D^4} [x_{34}(x_{12}^2 + x_{56}^2) - 3x_{12}x_{34} - x_{56}(x_{12}^2 + x_{34}^2)] ,$$

with

$$D \equiv D(\Lambda, \Lambda_0; p_1, p_2, p_3; x_1, \dots, x_6) \equiv x_{135}\Lambda^2 + x_{246}\Lambda_0^2 + \tilde{x}_{34}p_2^2 + \tilde{x}_{56}p_3^2 + 2x_{34}x_{56}p_2 \cdot p_3 .$$

$\Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}$  stays finite in the IR and UV limits i.e. finite when we take first  $\Lambda \rightarrow 0$  and then  $\Lambda_0 \rightarrow \infty$ .

## 2. The contracted triangle

For  $\Lambda = 0$  we obtain

$$p_{1\mu} \Gamma_{\mu\nu\rho}^{0, \Lambda_0} = \frac{2}{\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} \int d\mu_5 \frac{x_3 \Lambda_0^6}{[\tilde{x}_{25}p_2^2 + \tilde{x}_3 p_3^2 + 2x_{25}x_3 p_2 \cdot p_3 + x_{123}\Lambda_0^2]^3} \quad (99)$$

and in the UV limit, only one integral

$$\lim_{\Lambda_0 \rightarrow \infty} \lim_{\Lambda \rightarrow 0} (p_{1\mu} \Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}) = \frac{2}{\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} \int d\mu_6 \frac{x_3}{x_{123}^3}$$

survives. We find explicitly

$$\lim_{\Lambda_0 \rightarrow \infty} \lim_{\Lambda \rightarrow 0} (p_{1\mu} \Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}) = \frac{1}{6\pi^2} \epsilon_{\nu\rho\alpha\beta} p_{2\alpha} p_{3\beta} . \quad (100)$$

**3. Derivatives of the triangle** The first momentum derivatives are finite for  $\Lambda_0 \rightarrow \infty, \Lambda \rightarrow 0$ . In the second derivatives logarithmic divergences show up for exceptional momentum configurations when  $\Lambda \rightarrow 0$ . For example, we find

$$\delta_{\alpha\beta} \frac{\partial^2 \Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}}{\partial p_{2\alpha} \partial p_{2\beta}} \bigg|_{p_2=0, p_3 \neq 0}^{(div)} = \frac{1}{2\pi^2} \epsilon_{\mu\nu\rho\sigma} \frac{p_{3\sigma}}{p_3^2 + \Lambda^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) , \quad (101)$$

where  $\mu^2 > 0$  is an arbitrary momentum scale. Here the superscript is justified by the fact that

$$\delta_{\alpha\beta} \frac{\partial^2 \Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}}{\partial p_{2\alpha} \partial p_{2\beta}} \bigg|_{p_2=0, p_3 \neq 0} - \delta_{\alpha\beta} \frac{\partial^2 \Gamma_{\mu\nu\rho}^{\Lambda, \Lambda_0}}{\partial p_{2\alpha} \partial p_{2\beta}} \bigg|_{p_2=0, p_3 \neq 0}^{(div)} = p_{3\sigma} \epsilon_{\mu\nu\rho\sigma} f(\Lambda, \mu, p_3^2) \quad (102)$$

with

$$|f(\Lambda, \mu, p_3^2)| \leq \frac{20}{\pi^2 \sqrt{p_3^2(p_3^2 + 4\Lambda^2)}} \ln \left( 1 + \sqrt{\frac{p_3^2 + 4\Lambda^2}{p_3^2}} \right) .$$

Thus the r.h.s. is finite for all values of  $\Lambda$  if  $p_3 \neq 0$ .

## B On the Jacobian of regularized BRS-transformations

In this appendix we prove Lemma 4.1. We consider a finite cube of side length  $L$  in  $\mathbb{R}^4$ , and we expand the field variables in terms of plane waves

$$\left\{ e^{ik_n x} \mid k_n = (k_{n_0}, k_{n_1}, k_{n_2}, k_{n_3}), k_{n_i} = \frac{2\pi n_i}{L}, n_i \in \mathbb{Z} \right\}$$

thus imposing periodic boundary conditions <sup>16</sup>

$$\begin{aligned} A_\mu(x) &= \sum_{n \in \mathbb{Z}^4} A_{\mu,n} e^{ik_n x} , & \psi^i(x) &= \sum_{n \in \mathbb{Z}^4} \psi_{i,n} e^{ik_n x} , & \bar{\psi}^i(x) &= \sum_{n \in \mathbb{Z}^4} \bar{\psi}_{i,n} e^{ik_n x} , \\ c(x) &= \sum_{n \in \mathbb{Z}^4} c_n e^{ik_n x} , & \bar{c}(x) &= \sum_{n \in \mathbb{Z}^4} \bar{c}_n e^{ik_n x} . \end{aligned}$$

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<sup>16</sup>Our result also holds if we take antiperiodic boundary conditions for fermions and/or ghosts.

In the regularized theory we want to calculate the Jacobian  $J$  of the BRS transform of the integration measure for all field modes. This measure can be written as

$$\left( \prod_{\mu=0}^3 \prod_{n \in \mathbb{Z}^4} dA_{\mu,n} \right) \left( \prod_{j=1}^4 \prod_{n \in \mathbb{Z}^4} d\psi_{j,n} d\bar{\psi}_{j,n} \right) \left( \prod_{n \in \mathbb{Z}^4} dc_n d\bar{c}_n \right) \equiv \prod_{i=1}^{14} \prod_{n \in \mathbb{Z}^4} d\phi_{i,n} .$$

We write  $\Phi(x) = \{\phi_i(x), i = 1, \dots, 14\}$  for the set of all components of the fields of the theory with  $A_\mu = \phi_{\mu+1}$ ,  $\psi^i = \phi_{4+i}$ ,  $\bar{\psi}^i = \phi_{8+i}$ ,  $c = \phi_{13}$ ,  $\bar{c} = \phi_{14}$ .

$J$  is the determinant of a matrix  $((M_{ji})_{n'n'})$  built of blocks  $14 \times 14$ , with indices taking values in  $\mathbb{Z}^4$

$$(M_{ji})_{n'n'} = \frac{\partial \phi'_{i,n}}{\partial \phi_{j,n'}} .$$

We will call the matrix of elements  $\frac{\partial \phi'_{i,n}}{\partial \phi_{j,n'}}$  with  $i, j \in [1, 14]$ , the  $(n, n')$ -block of  $M$ .

The regularized BRS-transformations then induce the following changes of the field variables:

$$\begin{aligned} \sum_{n \in \mathbb{Z}^4} A'_{\mu,n} e^{ik_n x} &= \sum_{n \in \mathbb{Z}^4} A_{\mu,n} e^{ik_n x} - \sum_{n \in \mathbb{Z}^4} R_1^0 \int dy \sigma_{0,\Lambda_0}(x-y) ik_{n,\mu} c_n e^{ik_n y} \epsilon \\ \Rightarrow A'_{\mu,n} &= A_{\mu,n} - i R_1^0 \sigma_{0,\Lambda_0}(k_n) k_{n,\mu} c_n \epsilon , \\ \sum_{n \in \mathbb{Z}^4} \psi'_{i,n} e^{ik_n x} &= \sum_{n \in \mathbb{Z}^4} \psi_{i,n} e^{ik_n x} - i g R_2^0 \sum_{n,m \in \mathbb{Z}^4} \int dy \sigma_{0,\Lambda_0}(x-y) \psi_{i+2,n} c_m e^{i(k_n+k_m)y} \epsilon \\ \Rightarrow \psi'_{i,n} &= \psi_{i,n} - i g R_2^0 \sigma_{0,\Lambda_0}(k_n) \sum_{n_1+n_2=n} \psi_{i+2,n_1} c_{n_2} \epsilon , \end{aligned}$$

and similarly

$$\begin{aligned} \bar{\psi}'_{i,n} &= \bar{\psi}_{i,n} - i g R_3^0 \sigma_{0,\Lambda_0}(k_n) \sum_{n_1+n_2=n} \bar{\psi}_{i+2,n_1} c_{n_2} \epsilon , \\ c'_n &= c_n , \quad \bar{c}'_n = \bar{c}_n - \frac{i R_4^0}{\alpha} \sigma_{0,\Lambda_0}(k_n) k_n A_n \epsilon . \end{aligned}$$

We first study the diagonal blocks for which  $n = n'$ , and then the non-diagonal ones.

1. **Diagonal blocks** We obtain

$$\begin{aligned}
\frac{\partial A'_{\mu,n}}{\partial A_{\nu,n}} &= \delta_{\mu\nu} , \quad \frac{\partial A'_{\mu,n}}{\partial c_n} = -i R_1^0 k_{n,\mu} \sigma_{0,\Lambda_0}(k_n) \epsilon \\
\frac{\partial \psi'_{i,n}}{\partial \psi'_{j,n}} &= \delta_{ij} + i g R_2^0 \sigma_{0,\Lambda_0}(k_n) \delta_{i+2,j} c_{(0,0,0,0)} \epsilon , \quad \frac{\partial \psi'_{i,n}}{\partial c_n} = i g R_2^0 \sigma_{0,\Lambda_0}(k_n) \psi_{i+2,(0,0,0,0)} \epsilon , \\
\frac{\partial \bar{\psi}'_{i,n}}{\partial \bar{\psi}'_{j,n}} &= \delta_{ij} + i g R_3^0 \sigma_{0,\Lambda_0}(k_n) \delta_{i+2,j} c_{(0,0,0,0)} \epsilon , \quad \frac{\partial \bar{\psi}'_{i,n}}{\partial c_n} = i g R_3^0 \sigma_{0,\Lambda_0}(k_n) \bar{\psi}_{i+2,(0,0,0,0)} \epsilon , \\
\frac{\partial c'_n}{\partial c_n} &= 1 , \quad \frac{\partial \bar{c}'_n}{\partial \bar{c}_n} = 1 , \quad \frac{\partial \bar{c}'_n}{\partial A_{\mu,n}} = -i \frac{R_4^0}{\alpha} \sigma_{0,\Lambda_0}(k_n) k_{n,\mu} \epsilon .
\end{aligned}$$

All other coefficents are zero.

2. **Non-diagonal blocks**  $(n, n')$ . We have the relations

$$\begin{aligned}
\frac{\partial \psi'_{i,n}}{\partial \psi'_{j,n'}} &= \delta_{i+2,j} i g R_2^0 \sigma_{0,\Lambda_0}(k_n) c_{n-n'} \epsilon , \quad \frac{\partial \psi'_{i,n}}{\partial c_{n'}} = i g R_2^0 \sigma_{0,\Lambda_0}(k_n) \psi_{i+2,n-n'} \epsilon \\
\frac{\partial \bar{\psi}'_{i,n}}{\partial \bar{\psi}'_{j,n'}} &= \delta_{i+2,j} i g R_3^0 \sigma_{0,\Lambda_0}(k_n) c_{n-n'} \epsilon , \quad \frac{\partial \bar{\psi}'_{i,n}}{\partial c_{n'}} = i g R_3^0 \sigma_{0,\Lambda_0}(k_n) \bar{\psi}_{i+2,n-n'} \epsilon .
\end{aligned}$$

All other elements of this block are zero.

We then deduce an explicit expression for a general block  $(n, n')$ :

$$M_{n'n} = \delta_{n,n'} \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 & M_1 \\ 0 & \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_4 & 0 & 0 \\ M_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & M_3 & 0 & 0 & 0 \\ 0 & 0 & M_4 & 0 & 0 \\ 0 & M_5 & M_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$\begin{aligned}
M_1 &= -i \frac{R_4^0}{\alpha} \sigma_{0,\Lambda_0}(k_n) \epsilon , \\
M_2 &= -i R_1^0 \sigma_{0,\Lambda_0}(k_n) (k_{n,0} \ k_{n,1} \ k_{n,2} \ k_{n,3}) \epsilon , \\
M_{3/4} &= i g R_{2/3}^0 \sigma_{0,\Lambda_0}(k_n) c_{n-n'} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \epsilon , \\
M_5 &= i g R_2^0 \sigma_{0,\Lambda_0}(k_n) (\psi_{3,n-n'} \ \psi_{4,n-n'} \ \psi_{1,n-n'} \ \psi_{2,n-n'}) \epsilon , \\
M_6 &= i g R_3^0 \sigma_{0,\Lambda_0}(k_n) (\bar{\psi}_{3,n-n'} \ \bar{\psi}_{4,n-n'} \ \bar{\psi}_{1,n-n'} \ \bar{\psi}_{2,n-n'}) \epsilon .
\end{aligned}$$

To obtain a well-defined finite dimensional determinant we introduce an UV cutoff  $\Lambda_0$  through restricting the sum over Fourier modes to  $|k_n| \leq \Lambda_0$

$$\phi_{i,reg}(x) = \sum_{n \in \mathbb{Z}^4, |k_n| \leq \Lambda_0} \phi_{i,n} e^{ik_n x} ,$$

as stated in Lemma 4.1. Due to the cutoff the matrix  $\left( \frac{\partial \phi'_{i,n}}{\partial \phi_{j,n'}} \right)$  is finite-dimensional, and we can apply the usual formula for the determinant of an  $n \times n$ -matrix  $M$ , i.e.  $\det(M) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^n M_{i,\sigma(i)}$ . A nonvanishing element  $\alpha = (M_{ij})_{nn'} \neq 0, 1$  is of order  $\epsilon$ . Consider a contribution  $A$  to the determinant for which  $\alpha$  contributes. On the same line as  $\alpha$ , in the  $(n, n)$ -block, there is a unique nonvanishing coefficient  $\beta$  which equals 1.  $A$  is a multiple of a coefficient of the column of  $\beta$  (different from  $\beta$ ). But apart from  $\beta$ , the only non-zero elements in this column are of order  $\epsilon$ . Then  $A$  is zero because  $\epsilon^2 = 0$ .  $\square$

## C Comments on Fujikawa's argument

Fujikawa's argument [13], [14] links the chiral anomaly to the appearance of a Jacobian in the BRS transformation of the functional measure of integration. The argument is reproduced in many textbooks. From the mathematical point of view there are loopholes in this argument which we try to put into evidence, and it seems that the interpretation of Fujikawa's calculation, in particular in which sense precisely it may be related to the chiral anomaly, is unclear.

The arguments proceeds from a decomposition of the fermionic fields w.r.t. eigenbases  $\{\phi_n\}, \{\phi_n^\dagger\}$

$$\begin{aligned} \psi(x) &= \sum_n a_n \phi_n(x) \quad \Rightarrow \quad \psi'(x) \equiv e^{i\alpha(x)\gamma_5} \psi(x) = \sum_n a'_n \phi_n(x) , \\ \bar{\psi}(x) &= \sum_n b_n \phi_n^\dagger(x) \quad \Rightarrow \quad \bar{\psi}'(x) \equiv \bar{\psi}(x) e^{i\alpha(x)\gamma_5} = \sum_n b'_n \bar{\phi}_n^\dagger(x) , \end{aligned}$$

with

$$\begin{aligned} a'_m &= a_m + i \sum_n a_n \int dx \alpha(x) \phi_m^\dagger(x) \gamma_5 \phi_n(x) , \\ b'_m &= b_m + i \sum_n b_n \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 \phi_m(x) . \end{aligned}$$

Thus the Jacobian of this transformation is

$$\det \left( 1 + i \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 \phi_m(x) \right)^{-2} ,$$

because the variables  $a_n$ ,  $b_n$  are Grassmannian. Using the matrix relation  $\det(M) = \exp\{Tr \ln(M)\}$  with  $M = 1 + i \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 \phi_m(x)$ , and expanding the logarithm to first order in  $\alpha$ , we obtain the Jacobian

$$\prod_n \exp \left( -2i \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 \phi_n(x) \right) . \quad (103)$$

Using the plane wave basis of section B for the fermionic modes (on introducing cutoffs), we would conclude that this jacobian equals 1.

Fujikawa regularizes (103) in a way depending on the vector field which is viewed as a background field. In fact he introduces a smooth function  $f(x)$  such that  $f(0) = 1$ ,  $f(\infty) = 0$ , and writes a regularized Jacobian:

$$\prod_n \exp \left( -2i \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 f\left(\frac{\not{D}}{M^2}\right) \phi_n(x) \right) .$$

Here  $\not{D} = \gamma_\mu \partial_\mu - ieA_\mu$  is the covariant Dirac operator, and  $M < \infty$  is an UV regulator. The functions  $\phi_n$  are then supposed to be eigenfunctions of  $\not{D}$ . In this case the previous expression is well-defined only if the spectrum of  $\not{D}$  is discrete which generically will not be the case<sup>17</sup>. In the next step one passes to a plane wave basis using the relation

$$\lim_{M \rightarrow \infty} \sum_{n=1}^{\infty} \int dx \alpha(x) \phi_n^\dagger(x) \gamma_5 f\left(\frac{\not{D}}{M^2}\right) \phi_n(x) = \lim_{M \rightarrow \infty} \mathbf{tr} \int dx \alpha(x) \int_k e^{-ikx} \gamma_5 f\left(\frac{\not{D}}{M^2}\right) e^{ikx} , \quad (104)$$

where  $\mathbf{tr}$  indicates the spinor space trace. Applying the operator  $f\left(\frac{\not{D}}{M^2}\right)$  to  $e^{ikx}$ , and performing the change of variables  $k_\mu \rightarrow Mk_\mu$ , (104) becomes

$$\exp \left( -2i \lim_{M \rightarrow \infty} M^4 \mathbf{tr} \int dx \alpha(x) \int_k \gamma_5 f\left((ik_\mu + \frac{D_\mu}{M})^2 - \frac{ie}{4} [\gamma^\mu, \gamma^\nu] \frac{F_{\mu\nu}}{M^2}\right) \right) , \quad (105)$$

since  $\not{D}^2 = -D^2 - \frac{ie}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}$ . Expanding  $f$  around  $k_\mu$  and observing that only terms of order  $\geq -4$  in  $M$  and containing at least 4  $\gamma$ -matrices, survive for  $M \rightarrow \infty$ , (105) becomes

$$\begin{aligned} & \exp \left( \frac{ie^2}{8} \mathbf{Tr} (\gamma_5 [\gamma^\mu, \gamma^\nu] [\gamma^\rho, \gamma^\sigma]) \int dx \alpha(x) F_{\mu\nu}(x) F_{\rho\sigma}(x) \int_k f(k^2) \right) \\ &= \exp \left( 2ie^2 K \int dx \alpha(x) \tilde{F}^{\mu\nu}(x) F_{\mu\nu}(x) \right) , \end{aligned} \quad (106)$$

---

<sup>17</sup>a necessary condition would be that the field  $A_\mu(x)$  diverges for  $|x| \rightarrow \infty$ .

where  $K$  is a constant *depending on the function  $f$* . Choosing  $f(x) = e^{-x}$  one finds  $K = \frac{1}{16\pi^2}$ . The contribution of this Jacobian then gives rise to an anomalous term in the divergence of the axial current of the form

$$p_\mu \langle j_5^\mu(p) \rangle = 2i m \langle j_5(p) \rangle + \frac{ie^2}{8\pi^2} \langle (\tilde{F}_{\mu\nu} F^{\mu\nu})(p) \rangle \quad (\text{for fermions of mass } m) .$$

The result (106) has been obtained by introducing a background field dependent regulator for the fermions. Regularizing the fermions modes independently of this background would produce a trivial Jacobian as shown in App. B. The mathematical questions raised previously could be circumvented saying that what has been calculated is the short-time limit of the trace involving the diagonal part of the heat kernel  $K_t(x, y)$  of the operator

$$\exp\{-tD^2\}$$

in such a background field [23]:

$$\lim_{t \rightarrow 0} \mathbf{tr} \int dx \alpha(x) \gamma_5 K_t(x, x) . \quad (107)$$

But it is then not clear why this quantity should be directly related to the Jacobian of the chiral gauge transformation of the fermion fields.

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